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Remarks on 3 -prime near-ring involving * - involution

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Abstract

This work introduces the concept of * - involution in 3 - prime near-ring N together with its semi group ideal S and it establishes some results on N as well as S involving *- involution. In addition, examples are given to demonstrate the essentialities of 3- primeness in the hypothesis of our theorems. Finally, we conclude it with some open problems.

Keywords: Ring, near-ring, 3 -prime near-ring, * - involution, semi group ideal

1. Introduction

By a right near-ring we shall mean a non-empty set N endowed with two associative operations called addition $(+)$ and multiplication (denoted by $(+)$ and $(\cdot)$, respectively) satisfying the following conditions
(i) $(N\ ,+)$ is an additive group (not necessarily abelian)
(ii) $(N\ ,\ \cdot)$ is a semi group
(iii) Multiplication$(\cdot)$ distributes over addition$(+)$ from the right (denoted by

$(x+y)\cdot z = xz + yz\ \forall\ x,y,z\ \in N)$

A right near-ring N is said to be zero symmetric if $x \cdot 0 = 0\ \forall\ x\ \in N$(evoking that right distributive gives $0 \cdot x = 0$). Educing that N is said to be 3 prime near-ring, will have the property that $aNb = \{0\}$ for $a, b\ \in N$ implies $a = 0$ or $b =\ 0$. Normal subgroup S of $(N,+)$ is said to be an ideal of N if SN ⊆ S and $a(b+s) - ab \in S$ for $s \in S$ and $a, b \in N$.
A map $*: N \to N$ is said to be $*$ −involution if for $x, y\ \in N$, (i) $(x+y)^* = x^* + y^*$, (ii) $(xy)^* = x^*y^*$, (iii) $(x^*)^* = x$.
A near-ring N equipped with an $*$ −involution is called a near-ring with $*$ −involution or $*-$ near-ring. We refer the reader to the books of Clay [6], Meldrum [9] and Pilz [11] for the near-ring theory and its applications. Recall that a near-ring N is called 0 −prime if the product of any two of its ideals is non-zero. In addition, a near-ring N is called 3 −prime if for any non-zero $x, y \in N,\ xNy \neq \{0\}$ [7, 12]. Posner published his paper [13] in 1957; various authors have investigated the properties of derivations of prime and semi prime rings. Existence important ring theory tools [4], these outcomes are one of the sources of the developments of such theories as the theory of differential identities [8] and the theory of Hopf algebra action on rings [8], [10]. The study of derivations of near-rings was initiated by Bell and Mason in 1997 [2], but up to now only a few papers on 3-prime near-rings were published.
Bell, Boua, and Oukhtite [4] generalized some results known in this field involving the semi group ideal instead of entire near-rings. From these observations, one can ask a natural question "Can one apply the $*$ −involution on the structure of a $3-$prime near-ring N and its semi group ideal S? The aim of this paper is to give an affirmative answer to this question. In Section 2, we establish that a 3- prime near-ring N with $*$ −involution is an associative ring (or simply a ring). Section 3, devotes the result on semi group ideal of N with $*$ −involution becomes a ring. Also, we construct an example which establishes that our Theorems do not hold even for simple 0-prime near-rings with a right identity element.

2. On 3- prime near-ring with $*$ −involution

In this section, we establish the following result.

Theorem 2.1

Let N be a 3- prime near-ring with $*$ −involution. Then N

is a ring.

Proof

Assume that $*$ is an involution ($* -i$nvolution) on N. We claim that N is a ring.
We break the proof in two steps.

Step 1

We prove the multiplication on N satisfies left distributive law, that is

$$x(y+z) = xy + xz \quad \text{for all } x, y, z \in N \qquad (2.1)$$

Using the properties (iii), (ii) and (i) in the definition of $*$ $-$involution and right distributive law, we have

$x(y+z) = \left(\left(x(y+z)\right)^*\right)^* = ((y+z)^* x^*)^* = ((y^* + z^*)x^*)^* = (y^*x^* + z^*x^*)^* = ((xy)^* + (xz)^*)^* = ((xy)^*)^* + ((xz)^*)^* = xy + xz.$

This completes the proof of Step 1.

Step 2

We show that addition on N is abelian (viz: $(N,+)$ is abelian)
Replace x by $(y + z)$ and y and z by w in the relation (2.1) to get
$(y+z)(w+w) = (y+z)w + (y+z)w$
for any $w, y, z \in N$.

$(y+z)(w+w) = yw + zw + yw + zw$
for any $w, y, z \in N$ (2.2)

$(y+z)(w+w) = y(w+w) + z(w+w)$
for any $w, y, z \in N$.

$(y+z)(w+w) = yw + yw + zw + zw$
for any $w, y, z \in N$. (2.3)

Combining with the relations (2.2) and (2.3), we find that
$yw + zw + yw + zw = yw + yw + zw + zw$
for any $w, y, z \in N$.

$zw + yw = yw + zw$ for any $w, y, z \in N$.
$\left((z+y) - (y+z)\right)w = 0$ for all $w, y, z \in N$.

This implies that $\left((z+y) - (y+z)\right)N = \{0\}$ for all $y, z \in N$. (2.4)
In view of the result of Bell and Mason [2, Lemma 1.2 (i)], and relation (2.4), we have

$(z+y) - (y+z) = 0$. This implies $z + y = y + z$ for all $y, z \in N$.

Hence, $(N,+)$ is an additive abelian group. From Step 1 and Step 2, we see that a 3-prime near-ring N becomes a ring. ■

Remark 2.3

The following example shows that the condition of 3-prime near-ring in Theorem 2.1 is essential.

Example 2.4

Take a non-commutative near-ring M and define

$N = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} | \alpha, \beta \in M \right\}$, and a map $*: N \to N$ by $\begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$ for all $x, y \in M$.

Then $*$ is an involution ($* -$involution) on N, but N is neither a 3- prime near-ring nor a ring. For instance

For $* -$involution on N

Condition (i) $(x+y)^* = x^* + y^*$ and (iii) $(x^*)^* = x$, where $x = \begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & y_1 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, for all $x_1, x_2, y_1, y_2 \in S$, are straightforward.

(ii) $(xy)^* = \left[\begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & y_1 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]^* = \left[\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$y^*x^* = \begin{pmatrix} 0 & 0 & x_1 \\ 0 & 0 & x_2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & y_1 \\ 0 & 0 & y_2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Implies (xy)*= y* x*.

N is not a 3-prime near-ring

We have
$xNy = \begin{pmatrix} 0 & x_1 & x_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & y_1 & y_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, but $x \neq 0$ and $y \neq 0$.

From the above observations, one can easily see that N is not a ring. ■

3. Semi group ideal with $* -$ involution

We begin with the following definition

Definition 3.1

A non- empty subset S of N is called semi group right ideal (resp. semi group left ideal) of N if $SN \subseteq N$ (resp. $NS \subseteq N$); and S is said to be a semi group ideal if it is both a right semi group ideal as well as a left semi group ideal of N.

Example 3.2

Let $N = \{0, \alpha, \beta, \gamma\}$ with addition and multiplication tables defined as follows.
Taking $U = \{0, \alpha\}$, $V = \{0, \alpha, \beta\}$ and $W = (0, \alpha, \gamma\}$, then V, W are semi group right ideals of N and U is a semi group ideal of N.

$+$	0	α	β	γ
0	0	α	β	γ
α	α	0	γ	β
β	β	γ	0	α
γ	γ	β	0	α

$\cdot$	0	α	β	γ
0	0	0	0	0
α	0	0	α	α
β	0	α	β	β
γ	0	α	γ	γ

Theorem 3.1

Let N be a 3- prime near-ring and S a semi group ideal. In addition, if S admits $* -$ Involution then N is a ring.
In order to prove this theorem, we first state the result, due to Bell [2].

Fact 3.2

Let S be a non-zero semi group ideal of a 3-prime near-ring N with $x \in N$. given $xS = \{0\}$ or $Sx = \{0\}$ then $x = 0$.

Proof of Theorem 3.1

Keeping in mind the proof of Step 1 for entire 3-prime near-ring N, for the sake of convenience, we prove it for every a, b, c in semi group ideal S of N.

$$a(b+c) = \left(\left(a(b+c)\right)^*\right)^* = ((b+c)^* a^*)^* = (b^* a^* + c^* a^*)^* = (b^* a^*)^* + (c^* a^*)^* = a^{**} b^{**} + a^{**} c^{**}.$$

This implies that

$$a(b+c) = ab + ac \quad \text{for all } a, b, c \in S. \qquad (3.1)$$

Replacing mb for b and nb for c in (3.1), we get

$a(mb + nb) = amb + anb$ for all $a, b, c \in S$.
$[a(m+n) - (am + an)]b = 0$ for all $a, b, c \in S$ $m, n \in N$. But, for all $b \in S$, also

$$[a(m+n) - (am + an)]S = \{0\} \qquad (3.2)$$

Using Fact 3.2 and (3.2), we find that

$$l(m+n) = lm + ln \;\; \forall\, l, m, n \in N$$

Hence, the multiplication of N satisfies left distributive law, $(N, +)$ is an additive abelian group from Step 2 of Theorem 2.1. ■

Corollary 3.3

Let N be a 3-prime near-ring and S is a non- zero ideal of N. If S admits $* -$involution, then N is a ring.
Proof of the Corollary 3.3 follows immediately from Theorem 3.1. ■

Remark 3.4

We construct an example which shows that Theorem 3.1 does not hold even for simple 0-prime near-rings with a right identity element.

Example 3.5

Suppose that M be a linear space with a basis $B = \{e_M, e_2, e_3, \dots e_m\}$ over a field K of characteristic $\neq 2$. Define a multiplication $\cdot : M \times M \to M$ by the rule $m\,n = 0$ for all $m\,n \in M$ with $n \notin \{e_M, -e_M\}$ and $m\,e_M = m$, $m\,(-e_M) = -m$. It is easily seen that M is a right near-ring. Also M is a zero symmetric right near–ring with respect to this multiplication (See [1]).
Next, we show that M is a near-ring with the right identity e_M. Take a non-zero semi group ideal S of M. Let $e_M \in S$. Then M = M$e_M \subseteq S$. This is a contradiction. Thus $e_M \notin S$. If $n \in S$, then either $n + e_M \neq -e_M$ or $n + (-e_M) \neq e_M$. From the first case, it is easily seen that $e_M + n \neq e_M$. Thus $m\,(e_M + n) = 0$ for all $m \in M$. since S is a semi group ideal, we write $m = m(e_M + n) - m\,e_M \in S$, for all $m \in M$. This implies that $M \subseteq S$, a contradiction. Hence M is a right near-ring with identity e_M . Trivially, M is not a ring.

4. Open questions

In retrospect, we would like to open the questions for further studies as given below.
Question 1: Can the hypothesis that 3-prime be removed from the assumptions in Theorem 2.1 and Theorem 3.1?
Question 2: Can the hypothesis that semi group ideal be removed from the assumptions in Theorem 3.1?
Question 3: Can the hypothesis that $* -$involution be removed from the assumptions in Theorem 2.1 and Theorem 3.1?

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